$\boxed{8.5}$ Binomial Theorem

Pascal's Triangle

TED video

Secrets of Pascal Δ

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$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Binomial Coefficients $\quad {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

$$
\begin{array}{ccccccc}
 & & & \binom{1}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3}
\end{array}
\Rightarrow
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 2 & & 1 & \\
1 & & 3 & & 3 & & 1
\end{array}
\quad
\begin{array}{l}
n=0 \\
n=1 \\
n=2 \\
n=3
\end{array}
$$

Binomial Expansion

$$
\begin{aligned}
(x+y)^0 &= 1 \\
(x+y)^1 &= x+y \\
(x+y)^2 &= x^2 + 2xy + y^2 \\
(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\
(x+y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4
\end{aligned}
$$

Patterns? $(x+y)^n$

Coefficients = Pascal / Binomial Coefficients

Begin @ $\binom{n}{0}$ end @ $\binom{n}{n}$

First term begins at x^n decreasing to x^0

Second term begins at y^0 decreasing to y^n

Example 1 Determine the binomial expansion

Ⓐ $(x+y)^6$

$= \binom{6}{0}x^6y^0 + \binom{6}{1}x^5y^1 + \binom{6}{2}x^4y^2 + \binom{6}{3}x^3y^3 + \binom{6}{4}x^2y^4 +$
$\binom{6}{5}x^1y^5 + \binom{6}{6}x^0y^6$

$= \boxed{x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6}$

Ⓑ $(2a+b)^5$

$= \binom{5}{0}(2a)^5(b)^0 + \binom{5}{1}(2a)^4(b)^1 + \binom{5}{2}(2a)^3(b)^2 + \binom{5}{3}(2a)^2(b)^3 +$
$\binom{5}{4}(2a)^1(b)^4 + \binom{5}{5}(2a)^0(b)^5$

$= (1)(32a^5)(1) + (5)(16a^4)(b) + (10)(8a^3)(b^2) + (10)(4a^2)(b^3) +$
$(5)(2a)(b^4) + (1)(1)(b^5)$

$= \boxed{32a^5 + 80a^4b + 80a^3b^2 + 40a^2b^3 + 10ab^4 + b^5}$

Ⓒ $(y-3)^4$

$= \binom{4}{0}(y)^4(-3)^0 + \binom{4}{1}(y)^3(-3)^1 + \binom{4}{2}(y)^2(-3)^2 + \binom{4}{3}(y)^1(-3)^3 +$
$\binom{4}{4}(y)^0(-3)^4$

$= (1)(y^4)(1) + (4)(y^3)(-3) + (6)(y^2)(9) + (4)(y)(-27) + (1)(1)(81)$

$= \boxed{y^4 - 12y^3 + 54y^2 - 108y + 81}$

8.5 cont'd

Example 2 Find the indicated term

(A) 10th term of $(2c-3d)^{14}$ degree

degree $\rightarrow$ $14 = 5 + 9$ ← matches $\binom{14}{9}$

$$\binom{14}{9}(2c)^5(-3d)^9$$

one less than the term

$${}_{14}C_9\, 2^5 c^5 (-3)^9 d^9 = \underline{2002} \cdot \underline{32} c^5 (\underline{-19683}) d^9$$

combine

$$\boxed{-1260971712\, c^5 d^9}$$

rth term of Binomial Expansion of $(x+y)^n$

$$\binom{n}{r-1} x^{n-(r-1)} y^{r-1}$$