5.5 Exponential and logarithmic Equations

Common Base Property

If $b^x = b^y$, then $x = y$

If $\log_b x = \log_b y$, then $x = y$

Solving Exponential & Logarithmic Equations

Hints: ① Change forms

$$B^P = A \iff \log_B A = P$$

② Look for common bases

③ Take the log or ln of both sides

Remember rules:

- $\ln e = 1$, $\log 10 = 1$, $\log_a a = 1$
- $\ln 1 = 0$ $\log 1 = 0$ $\log_a 1 = 0$
- $\ln e^x = x$ $\log 10^x = x$ $\log_a a^x = x$

Properties

- $\log(xy) = \log x + \log y$
- $\log\left(\frac{x}{y}\right) = \log x - \log y$
- $\log x^a = a \log x$

Change of Base

$$\log_a x = \frac{\log x}{\log a}$$

Directions:

Exact answer ⇒ Do not round

i.e $\ln 8$ ⇒ leave as $\boxed{\ln 8}$

Round to nearest thousandth ⇒ use calculator

i.e $\ln 8 = 2.0794415\ldots = \boxed{2.079}$

Example 1 Solve. Express solutions in exact form. Do not use a calculator.

② $2 \cdot \frac{1}{2} e^x = 13 \cdot 2$

$e^x = 26$ — Take ln of both sides

$\ln e^x = \ln 26$ — move x to front of lne

$x \ln e = \ln 26$ — $\ln e = 1$

$\boxed{x = \ln 26}$

Alternate method:

$2 \cdot \frac{1}{2} e^x = 13 \cdot 2$

$e^x = 26$ — Change to logarithm: $B^P = A \Rightarrow \log_B A = P$

$\log_e 26 = x$

$\boxed{\ln 26 = x}$

④ $5(10^{3x}) - 4 = 6$ — Add 4 to both sides

$\quad +4 \quad +4$

$\frac{5(10^{3x})}{5} = \frac{10}{5}$ — Divide both sides by 5

$10^{3x} = 2$ — Take log of both sides

$\log 10^{3x} = \log 2$ — Use The power rule

$3x \log 10 = \log 2$ — $\log 10 = 1$

$\frac{3x}{3} = \frac{\log 2}{3}$ — Divide both sides by 3

$\boxed{x = \frac{\log 2}{3}}$

$\boxed{5.5}$ cont'd

Example 2 Solve. Express as (a) Exact (b) nearest thousandth

(10) $5^x = 13$

log both sides

$\log 5^x = \log 13$

$\dfrac{x \log 5}{\log 5} = \dfrac{\log 13}{\log 5}$

(a) $\boxed{x = \log_5 13}$

(b) $\boxed{x = 1.594}$

Change forms

$B^P = A \Rightarrow \log_B A = P$

(a) $\boxed{\log_5 13 = x}$

(b) $\boxed{1.594 = x}$

(12) $\left(\frac{1}{3}\right)^x = 6$

$\log \frac{1}{3}^x = \log 6$

$\dfrac{x \log\left(\frac{1}{3}\right)}{\log \frac{1}{3}} = \dfrac{\log 6}{\log \frac{1}{3}}$

(a) $\boxed{x = \log_{\frac{1}{3}} 6}$

(b) $\boxed{x = -1.631}$

(a) $\boxed{\log_{\frac{1}{3}} 6 = x}$

(b) $\boxed{x = -1.631}$

(16) $2^{x+3} = 5^x$

$\log 2^{x+3} = \log 5^x$

$(x+3)\log 2 = x \log 5$

$x \log 2 + 3 \log 2 = x \log 5$

$-x\log 2 \qquad -x \log 2$

$3 \log 2 = x \log 5 - x \log 2$

$\log 2^3 = x(\log 5 - \log 2)$

$\dfrac{\log 8}{\log \frac{5}{2}} = \dfrac{x\left(\log \frac{5}{2}\right)}{\log \frac{5}{2}}$

$\boxed{\log_{\frac{5}{2}} 8 = x}$ $\quad$ $\boxed{x = 2.269}$

Change forms

$\log_2 5^x = x + 3$

$x \log_2 5 = x + 3$

$-x \qquad -x$

$x \log_2 5 - x = 3$

$\dfrac{x(\log_2 5 - 1)}{\log_2 5 - 1} = \dfrac{3}{\log_2 5 - 1}$

$\boxed{x = \dfrac{3}{\log_2 5 - 1}}$

$\boxed{x = 2.269}$

HW

(17) $6^{x+1} = 4^{2x-1}$

$\log 6^{x+1} = \log 4^{2x-1}$ — Log both sides

$(x+1)\log 6 = (2x-1)\log 4$ — Bring down the power

$x\log 6 + \log 6 = 2x\log 4 - \log 4$ — Distribute

$\log 6 + \log 4 = 2x\log 4 - x\log 6$ — Get x's to one side

$\log 6 + \log 4 = x(2\log 4 - \log 6)$ — Factor

$\log(6 \cdot 4) = x(\log 4^2 - \log 6)$ — Use log properties

$\log(24) = x(\log \frac{16}{6})$ — Simplify

$\frac{\log(24)}{\log \frac{8}{3}} = \frac{x(\log \frac{8}{3})}{\log \frac{8}{3}}$ — Divide by $\log \frac{8}{3}$

$\boxed{\log_{\frac{8}{3}} 24 = x}$ Bonus

or $\frac{\log 6 + \log 4}{\log 16 - \log 6} = x$ ok

$\boxed{x = 3.240}$

Book $\frac{-\log 6 - \log 4}{\log 6 - 2\log 4}$

(20) $3^x = -4$

$\log_3(-4) = x$

Domain ERROR

undefined; $\varnothing$

(22) $e^{.5x} = 3^{1-2x}$ — Use ln since $\ln e = 1$

$\ln e^{.5x} = \ln 3^{1-2x}$ — Take ln of both sides

$.5x \ln e = (1-2x)\ln 3$

$.5x = \ln 3 - 2x\ln 3$

$.5x + 2x\ln 3 = \ln 3$

$\frac{x(.5+2\ln 3)}{.5+2\ln 3} = \frac{\ln 3}{.5+2\ln 3}$

$\boxed{x = \frac{\ln 3}{.5+\ln 9}}$

$\boxed{5.5}$ cont'd

(30) $3(1.4)^x - 4 = 60$

$\quad\quad +4 \quad +4$

$\frac{3(1.4)^x}{3} = \frac{64}{3}$

$1.4^x = \frac{64}{3}$

$\boxed{\log_{1.4}\left(\frac{64}{3}\right) = x}$ → $x = \frac{\log\left(\frac{64}{3}\right)}{\log 1.4}$

$\boxed{x = 9.095}$

Example 3 Solve each equation. Give answer in exact form

(34) $\frac{3\log x}{3} = \frac{2}{3}$

$\log x = \frac{2}{3}$

$\boxed{10^{\frac{2}{3}} = x}$

or

$\boxed{\sqrt[3]{10^2} = x}$

(36) $\ln 2x = 5$

$\frac{e^5}{2} = \frac{2x}{2}$

$\boxed{\frac{e^5}{2} = x}$

(40) $\log_5(8-3x) = 3$

$5^3 = 8 - 3x$

$\frac{5^3 - 8}{-3} = \frac{-3x}{-3}$

$\frac{5^3-8}{-3} = x$

$\boxed{-39 = x}$

(46) $\log x + \log(3x-13) = 1$

$\log(x(3x-13)) = 1$

$\log(3x^2 - 13x) = 1$

$10^1 = 3x^2 - 13x$

$0 = 3x^2 - 13x - 10$

$0 = 3x^2 - 15x + 2x - 10$

$0 = 3x(x-5) + 2(x-5)$

$0 = (3x+2)(x-5)$

$\boxed{x = -\frac{2}{3}}$ $\boxed{x = 5}$

(52) $\log_2(2x) + \log_2(x+2) = \log_2 16$

$\log_2(2x(x+2)) = \log_2 16$

$\log_2(2x^2+4x) = \log_2 16$

$2x^2+4x = 16$

$2x^2+4x-16=0$

$x^2+2x-8=0$

$(x+4)(x-2)=0$

$\boxed{x=-4}$ $\boxed{x=2}$

Example 4 Solve for the indicated variable X

(64) $p = a + \frac{k}{\ln x}$

$p - a = \frac{k}{\ln x}$

$\frac{\ln x(p-a)}{p-a} = \frac{k}{p-a}$

$\ln x = \frac{k}{p-a}$

$\boxed{e^{\frac{k}{p-a}} = x}$

Example 5 Solve. Give exact answers

(76) $e^{2x} - 8e^x + 15 = 0$

let $u = e^x$

$u^2 - 8u + 15 = 0$

$(u-3)(u-5) = 0$

$u=3 \quad u=5$

$e^x = 3 \quad e^x = 5$

$\boxed{\ln 3 = x}$ $\boxed{\ln 5 = x}$

5.5 cont'd

(80) $\frac{1}{4}e^{2x} + 2e^{x} = -3$

$4\left(\frac{1}{4}e^{2x} + 2e^{x} + 3 = 0\right)$

$e^{2x} + 8e^{x} + 12 = 0$

let $u = e^{x}$

$u^2 + 8u + 12 = 0$

$(u+6)(u+2) = 0$

$e^{x} = -6 \quad e^{x} = -2$

$\ln(-6) = x \quad \ln(-2) = x$

Domain Error

$\boxed{\text{no solution}}$

(86) $2(\ln x)^2 + 9(\ln x) = 5$

let $u = \ln x$

$2u^2 + 9u - 5 = 0$

$2u^2 + 10u - u - 5 = 0$

$2u(u+5) - 1(u+5) = 0$

$(2u-1)(u+5) = 0$

$2u - 1 = 0 \quad u + 5 = 0$

$u = \frac{1}{2} \quad u = -5$

$\ln x = \frac{1}{2} \quad \ln x = -5$

$\boxed{e^{\frac{1}{2}} = x} \quad \boxed{e^{-5} = x}$

Time Permit: Graph

Example 6 Solve $f(x) = 0$ analytically. Use a graph to solve $f(x) < 0$ and $f(x) > 0$

(88) $f(x) = -3e^{x} + 7$

$0 = -3e^{x} + 7$

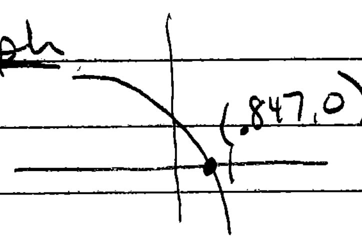
$-7 = -3e^{x}$

$\frac{7}{3} = e^{x}$

$\ln\frac{7}{3} = x$

$\boxed{.847 = x}$

Graph



$f(x) < 0 \quad x > .847 \quad (.847, \infty)$

$f(x) > 0 \quad x < .847 \quad (-\infty, .847)$

94) $f(x) = 9\log_3(3x) - 18$

$0 = 9\log_3(3x) - 18$

$\dfrac{18}{9} = \dfrac{9\log_3(3x)}{9}$

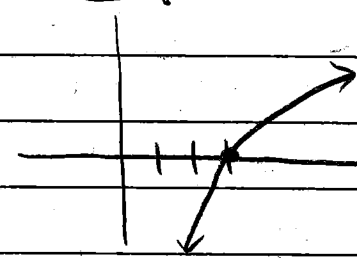
$2 = \log_3(3x)$

$3^2 = 3x$

$\dfrac{9}{3} = \dfrac{3x}{3}$

$\boxed{x = 3}$

Graph



$f(x) < 0 \quad x < 3 \quad (-\infty, 3)$

$f(x) > 0 \quad x > 3 \quad (3, \infty)$